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Renormalization-Group Functions and Nonuniversal Critical Behaviour.

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Abstract. – The method of Borel resummation is used to obtain an accurate representation of renormalization-group functions of the n-vector model within the minimal renormalization scheme. This permits to relate the nonuniversal critical behaviour of different quantities well away from criticality. The results can be used to estimate the range of applicability of the n-vector model. Application to the λ-transition of ^{4}He explains the measured effective exponent of the superfluid density along the λ-line.

A theory of critical phenomena that is applicable well beyond the asymptotic critical region encounters the problem of how to deal with the *nonuniversal* properties of the specific system under consideration. Within a renormalization-group (RG) description the fundamental nonuniversal parameters are the effective renormalized couplings which are the solutions of the RG flow equations. Alternatively it has been shown [1, 2] that these couplings can be determined rather directly in terms of experimental values of the specific heat without integration of the RG flow equations. So far this approach has been applied [3] only within a low-order perturbation calculation. It has, therefore, been suggested [2] to improve the accuracy of this approach by taking advantage of high-order perturbation theory and the Borel resummation method [4, 5]. In the present letter we report on the results of a corresponding study of the standard n-vector model including an application to the λ-transition of ^{4}He. Our main results are i) an accurate representation of the RG functions for $n = 1, 2, 3$ within the minimal subtraction scheme [6], and ii) an explanation of the observed [7] temperature and pressure dependence of the effective exponent of the superfluid density below the λ-line of ^{4}He ($n = 2$).

We start out from the usual Landau-Ginzburg-Wilson Hamiltonian in d-dimensions

$$H_{\varphi} = \int d^dx \left[\frac{1}{2} r_0 \varphi_0^2 + \frac{1}{2}(\nabla\varphi_0)^2 + u_0 \varphi_0^4\right], \tag{1}$$

where $\varphi_0(x)$ is an n-component vector field. In the application of our results we shall also refer to the extended Hamiltonian

$$H = \int d^d x \left[\frac{1}{2} \tau_0 \varphi_0^2 + \frac{1}{2} (\nabla \varphi_0)^2 + \tilde{u}_0 \varphi_0^4 + \frac{1}{2} \chi_0^{-1} m_0^2 + \gamma_0 m_0 \varphi_0^2 - h_0 m_0 \right] \tag{2}$$

with $\tilde{u}_0 - \chi_0 \gamma_0^2/2 = u_0$, $\tau_0 + 2\gamma_0 h_0 \chi_0 = r_0$, which is of some advantage in describing the specific heat [2]. We shall treat the asymptotic and nonasymptotic critical behaviour of these models by means of the field-theoretic RG approach within the minimal subtraction scheme [6]. We prefer to use this renormalization scheme because of its simplicity and its substantial advantage in the application to critical dynamics [8].

Using the notation of ref. [2] we introduce the renormalized quantities $u = \mu^{-\varepsilon} Z_u^{-1} Z_\varphi^2 A_d u_0$, $r = Z_r^{-1} r_0$, $\varphi = Z_\varphi^{-1/2} \varphi_0$, and the corresponding RG functions

$$\beta_u(u, \varepsilon) = (\mu \partial_\mu u)_0 = -\varepsilon u + \tilde{\beta}(u) , \tag{3}$$

$$\zeta_r(u) = (\mu \partial_\mu \ln Z_r^{-1})_0 , \tag{4}$$

$$\zeta_\varphi(u) = (\mu \partial_\mu \ln Z_\varphi^{-1})_0 , \tag{5}$$

with $\varepsilon = 4 - d$. For an accurate description of the *nonasymptotic* critical behaviour it is crucial to have available a precise representation of these functions in the complete range $0 \leq u \leq u^*$, where u^* is the fixed-point value. So far this information has not yet been given in the existing literature on the n-vector model within the minimal subtraction scheme. We shall provide this information in this letter for $n = 1, 2, 3$ using the well-known Borel resummation method [4].

The functions $\tilde{\beta}(u)$, $\zeta_r(u)$ and $\zeta_\varphi(u)$ will be denoted by $f_i(u)$, $i = 1, 2, 3$, respectively. They have a perturbation expansion

$$f_i(u) = \sum_k f_i^{(k)} u^k , \tag{6}$$

whose coefficients $f_i^{(k)}$ have been calculated [5, 9] up to order $k = 6$ for $i = 1$ and $k = 5$ for $i = 2, 3$. In addition their large-order behaviour

$$f_i^{(k)} \sim k!(-a)^k k^{b_i} c_i [1 + O(1/k)] , \tag{7}$$

is known [10], where $a = 4!$, $b_1 = b_2 = 3 + n/2$, $b_3 = 2 + n/2$. (The constants c_i are not of interest in the following.) Using a Borel transformation and a conformal mapping one obtains, instead of (6), the expansion [4]

$$f_i(u) = \sum_k F_i^{(k)} I_i^{(k)}(u) \tag{8}$$

with

$$I_i^{(k)}(u) = \int_0^\infty t^b \exp[-t] \frac{x(ut)^k}{[1 - x(ut)]^{\alpha_i}} dt , \tag{9}$$

where $x(ut) = (y - 1)/(y + 1)$ with $y = (1 + aut)^{1/2}$. Since the coefficients $F_i^{(k)}$ can be expressed in terms of $f_i^{(j)}$ with $j \leq k$, the series (8) is known up to order $k = 5$ (or $k = 6$). The sum (8) truncated at order $k = L$ will be denoted by $S_i^{(L)}$. The parameters α_i have been determined by requiring the fastest convergence of the sequence of $S_i^{(L)}$ up to $L = 5$ (or

TABLE I. – *Parameters* α_i *of eq.* (9) *and* a_i *of eqs.* (10) *and* (11). *The fixed-point values* $u^* = \bar{\beta}(u^*)$ *are obtained from fig.* 1 (*estimated error* ± 0.001); *last column contains the fixed-point values* $g_0/12$ *of ref.* [5].

	α_1	α_2	α_3	a_1	a_2	a_3	u^*	$g_0/12$
$n=1$	1.4	1.5	1.5	3075	30390	37.5	0.0405	0.0407
$n=2$	1.5	1.5	1.5	4851	57309	74.6	0.0362	0.0363
$n=3$	1.6	1.5	1.8	6103	66553	111	0.0328	0.0328

$L=6$). With $b = b_i + 1.5$ this leads to the values of α_i listed in table I. We have verified that these values depend only weakly on the choice of b. The functions $f_i(u)$ are finally obtained by an appropriate extrapolation of $\bar{S}_i^{(L)}$. In fig. 1 $\beta_u(u, 1) = -u + \bar{\beta}(u)$ is plotted for $n = 1, 2, 3$ (with error bars for the example of $n = 2$). The fixed-point values $u^* = \bar{\beta}(u^*)$ at $\varepsilon = 1$ (table I) are very close to those of the ε-expansion of ref. [5] (our estimate of the error is about 2-3 times larger). In the range $0 \leqslant u \leqslant u^*$ our results for $\zeta_r(u)$ and $\zeta_{\varphi}(u)$ can be approximated, within the accuracy of our calculation, by the simple representations

$$\zeta_r(u) = (n+2)(4u - 40u^2) + a_1 u^3 - a_2 u^4 , \tag{10}$$

$$\zeta_{\varphi}(u) = -8(n+2)\, u^2 + a_3 u^3 , \tag{11}$$

with constants a_i listed in table I. The ensuing critical exponents $\nu^{-1} = 2 - \zeta_r(u^*)$ and $\eta = -\zeta_{\varphi}(u^*)$ agree with the known values [11]. Further details of our calculation are given in ref. [12].

We briefly illustrate the significance of our results by an application to the λ-line of ^{4}He in the following steps i)-iv):

i) It suffices to use the experimental [13] specific heat $\overset{0}{C}{}^+$ in a limited temperature range above T_λ in order to determine the effective renormalized couplings $u(t)$ and $\gamma(t)$ related to the bare couplings u_0 and γ_0 of (1) and (2). Using the relative temperature

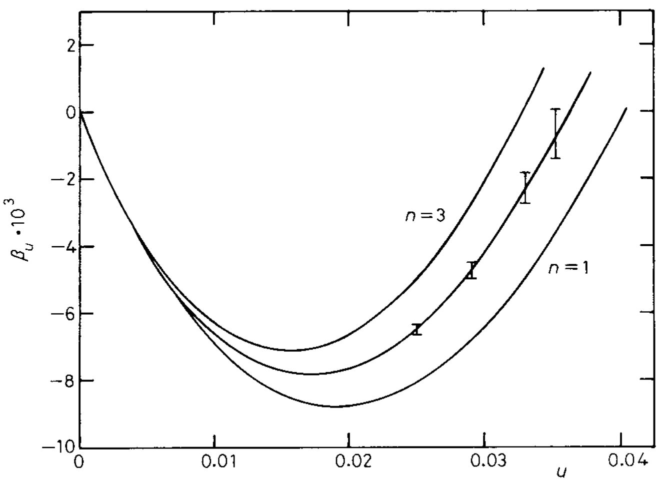

Fig. 1. – $\beta_u(u, 1) = -u + \bar{\beta}(u)$, eq. (3), for $n = 1, 2, 3$. Error bars are given for $n = 2$.

$t = (T - T_\lambda)/T_\lambda$ as a flow parameter, we obtain the RG flow equations [2] in the form

$$t\partial_t u(t) = \beta_u(\hat{u}, 1)/(2 - \zeta_r(\hat{u})) , \tag{12}$$

$$t\partial_t \gamma(t)^2 = \hat{\gamma}^2(-1 + 2\zeta_r(\hat{u}) + 4\hat{\gamma}^2)/(2 - \zeta_r(\hat{u})) , \tag{13}$$

with $\hat{u} \equiv u(t)$, $\hat{\gamma} \equiv \gamma(t)$. We treat the initial values $u(t_0) = u$ and $\gamma(t_0)^2 = \gamma^2$, at some arbitrary reference point t_0, as two adjustable parameters. On the basis of eq. (4.20) of ref. [2], they can be determined from the experimental values of $d(\ln \mathcal{C}^+)/d(\ln t)$ at two points, say $t = 10^{-3.5}$ and $t = 10^{-4}$, for a given pressure P. Integration of (12) and (13) then yields $u(t)$ and $\gamma(t)^2$ for arbitrary t (at this pressure). In fig. 2 the results are shown for SVP and 28 bar (solid curves). For an alternative determination see iv).

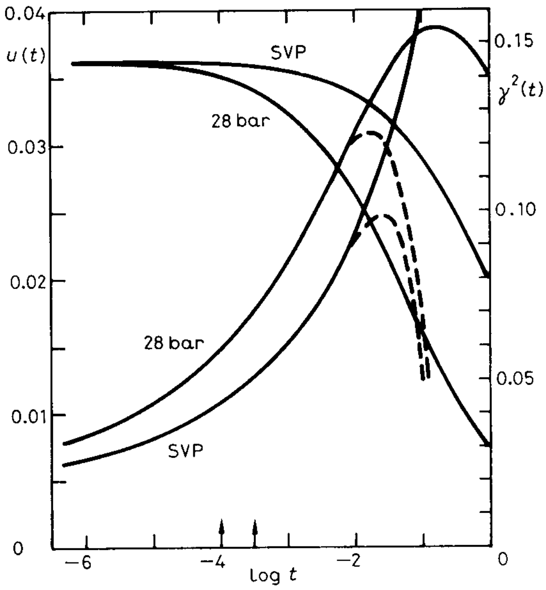

Fig. 2. – Effective couplings $u(t)$ and $\gamma(t)^2$ *vs.* reduced temperature $t = (T - T_\lambda(P))/T_\lambda(P)$ for SVP and 28 bar calculated by integrating (12) and (13) (solid curves). The arrows indicate the region where specific-heat data have been used. The dashed curves are obtained from eq. (6.8) of ref. [2]. Compare fig. 1 of Dohm (ref. [3]) and fig. 1 of Tam and Ahlers (ref. [3]).

ii) On the basis of (1) we have calculated the effective exponent ζ_{eff} of the superfluid density ρ_s below T_λ. In $d = 3$ we find the general form

$$\frac{\mathrm{d}\ln \rho_s(|t|)}{\mathrm{d}\ln |t|} \equiv \zeta_{\text{eff}}(|t|) = (2 - \zeta_r(\bar{u}))^{-1}\left[1 + \beta_u(\bar{u}, 1)\frac{\mathrm{d}\ln G(\bar{u})}{\mathrm{d}\bar{u}}\right] , \tag{14}$$

where in one-loop order

$$G(\bar{u}) = (8\bar{u})^{-1} + \tfrac{1}{3} + O(\bar{u}) \tag{15}$$

with $\bar{u} \equiv u(2|t|)$. We see that ζ_{eff} depends most sensitively on $\zeta_r(\bar{u})$ and $\beta_u(\bar{u}, 1)$. We expect that, because of the smallness of $\bar{u}$, the approximation (15) is sufficiently accurate for the

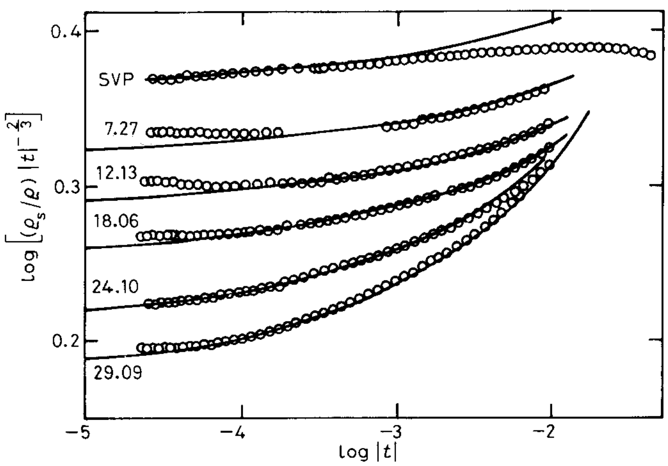

Fig. 3. – High-resolution plot of the experimental superfluid fraction ρ_s/ρ after fig. 2 of ref. [7] (circles). The numbers indicate the pressures in bar. The temperature dependence of the solid curves is the theoretical prediction derived from eqs. (14) and (15) with $\bar{u}$ determined via step i) (fig. 2).

present purpose. In fig. 3 the original data of Greywall and Ahlers [7] are shown. Integration of (14) and adjustment of a (temperature independent) amplitude $A(P)$ yields the corresponding representation of our theoretical result (solid curves in fig. 3). Most significant is the *nonuniversal curvature* of the data at higher pressure which is satisfactorily explained by our ζ_{eff} *without adjustments.*

iii) The expressions for the renormalized specific heat $C^{\pm}$ above and below T_λ in terms of $u(t)$ and $\gamma(t)^2$ have been given in ref. [2]. We only need the adjustment of the absolute amplitude of C^+ at one single point (say at $t = 10^{-4}$) above T_λ in order to plot the complete theoretical specific heat above and below T_λ *outside the range where the adjustments have been made.* We find good agreement between the data [13] and our theory in the range $t < 10^{-2}$ and $-t < 10^{-3}$ for all P.

iv) We have also determined $u(t)$ and $\gamma(t)^2$ without integration of the flow equations (according to eqs. (6.1) and (6.8) of ref. [2]). For $t < 10^{-2}$ the resulting $u(t)$ and $\gamma(t)$ agree with those obtained in i) above (fig. 2). The deviations of $\gamma(t)^2$ for $t > 10^{-2}$ are shown by the dashed curves in fig. 2. They can serve as an effective parametrization of the experimental specific heat in the precritical region even beyond the range of applicability of the RG flow eq. (13). For an important application see ref. [14].

In summary we have presented an accurate representation of the RG functions of the n-vector model and have shown that the general concept of a theory of nonuniversal critical phenomena outlined in ref. [1] and [2] works successfully in case of the λ-transition of ^{4}He. In particular we have explained the nonuniversal effective exponent of the superfluid density in the entire experimental range below T_λ using only a few data points of the specific heat above T_λ. A further improvement of the theory would consist in a more accurate (high-order) calculation of the function $G(u)$, eq. (15), and of the specific-heat functions $F_\pm$ of ref. [2]. A more detailed version of the present paper including a comparison with the different renormalization scheme of ref. [15] will be published elsewhere.

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