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FINITE-SIZE EFFECTS IN THE φ^4 FIELD AND LATTICE THEORY ABOVE THE UPPER CRITICAL DIMENSION

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We demonstrate that the standard $O(n)$ symmetric φ^4 field theory does not correctly describe the leading finite-size effects near the critical point of spin systems with periodic boundary conditions on a d-dimensional lattice with $d > 4$. We show that these finite-size effects require a description in terms of a lattice Hamiltonian. For $n \to \infty$ and $n = 1$, explicit results are given for the susceptibility and for the Binder cumulant. They imply that these quantities do not have the universal properties predicted previously and that recent analyses of Monte Carlo results for the five-dimensional Ising model are not conclusive.

Keywords: Finite-Size Scaling; φ^4 Field Theory; Upper Critical Dimension; Five-Dimensional Ising Model.

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The effect of a finite geometry on systems near phase transitions is of basic interest to statistical physics and elementary particle physics. In both areas the φ^4 Hamiltonian

$$H = \int_V d^d \left[\frac{1}{2} r_0 \varphi^2 + \frac{1}{2}(\nabla\varphi)^2 + u_0(\varphi^2)^2\right] \tag{1}$$

for an n-component field $\varphi(\mathbf{x})$ in a finite volume V plays a fundamental role.[1] For simplicity we consider a d-dimensional cube, $V = L^d$, with periodic boundary conditions,

$$\varphi(\mathbf{x}) = L^{-d} \sum_{\mathbf{k}} \varphi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \quad . \tag{2}$$

The summation runs over discrete $\mathbf{k}$ vectors with components $k_j = 2\pi m_j/L$, $m_j = 0, \pm 1, \pm 2, \ldots, j = 1, 2, \ldots, d$, in the range $-\Lambda \le k_j < \Lambda$ with a finite cutoff Λ.

It is generally believed that the leading finite-size effects near the critical point of d-dimensional systems can be described by H both for $d \le d_u$ and for $d > d_u$

where $d_u = 4$ is the upper critical dimension. While finite-size effects are generally expected to depend on the geometry and boundary conditions[2] it appears to be well established that for $d < d_u$ and for cubic geometry with periodic boundary conditions there exist universal finite-size scaling functions that are independent of u_0 and of Λ.[3-6] Since for $d > 4$ the bulk critical behavior is mean-field like, it is plausible that the leading finite-size effects for $d > 4$ appear to be describable in terms of a simplified Hamiltonian[3,4]

$$H_0(\Phi) = L^d \left[\frac{1}{2} r_0 \Phi^2 + u_0 (\Phi^2)^2\right] \tag{3}$$

involving only the homogeneous fluctuations of the lowest ($\mathbf{k} = 0$) mode $\varphi_0 = L^d \Phi$,

$$\Phi = L^{-d} \int_V d^d x \varphi(\mathbf{x}). \tag{4}$$

Based on the statistical weight $\exp[-H_0(\Phi)]$, universal results have been derived[3] for all moments

$$\langle (\Phi^2)^p \rangle = (u_0 L^d)^{-p/2} f_{2p}(u_0^{1/2} t L^{d/2}) \tag{5}$$

with universal *single-variable* finite-size scaling functions $f_{2p}(x), t = (T - T_c)/T_c$. In particular, for $p = 1$, the power law

$$\chi_c = u_0^{-1/2} f_2(0) L^{d/2} \tag{6}$$

was predicted for the susceptibility at T_c.[3] For the case $n = 1$, the lowest-mode predictions have been compared with Monte Carlo (MC) data for the five-dimensional Ising model.[7-10] Although disagreements were noted in Ref. 7, subsequent analyses[8-10] based on H claimed to reconcile the MC data with the lowest-mode predictions.

In this paper[11] we shall demonstrate that the lowest-mode approach fails for $d > 4$ and that the leading finite-size effects of spin systems with periodic boundary conditions on a d-dimensional lattice with $d > 4$ are not correctly described by H. We show that the predicted[3] universal structure, Eq. (5), and the power law $\chi_c \sim L^{d/2}$ are not valid for the ϕ^4 model, Eq. (1), for $d > 4$. These unexpected findings imply that recent analyses of the MC data[7-10] in terms of the continuum φ^4 theory are not conclusive. Instead of the φ^4 field-theoretic Hamiltonian we shall use a φ^4 lattice Hamiltonian to predict the analytic asymptotic form of the size and temperature dependence of the Binder cumulant for $d > 4$ dimensions. This analytic result can be compared with MC data for the five-dimensional Ising model.

We shall prove our claims first in the large-n limit where a saddle point approach[1] can be employed. Our proof is not based on the renormalization group. We have extended the saddle point approach to the finite system to derive the order-parameter correlation function

$$\chi = (1/n) L^d \langle \Phi^2 \rangle = (1/n) \int_V d^d x \langle \varphi(\mathbf{x}) \varphi(0) \rangle \tag{7}$$

with the statistical weight $\exp(-H)$. In the limit $n \to \infty$ at fixed $u_0 n$ we have found the exact result

$$\chi^{-1} = r_0 + 4u_0 n L^{-d} \sum_{\mathbf{k}} (\chi^{-1} + \mathbf{k}^2)^{-1}. \tag{8}$$

For $T \geq T_c$, χ can be interpreted as the susceptibility (per component) of the finite system. In the bulk limit the standard equation for the bulk susceptibility χ_b for $T \geq T_c$ is recovered from Eq. (8) as

$$\chi_b^{-1} = r_0 + 4u_0 n \int_{\mathbf{k}} (\chi_b^{-1} + \mathbf{k}^2)^{-1}, \tag{9}$$

where $\int_{\mathbf{k}}$ stands for $(2\pi)^{-d} \int d^d k$ with a finite cutoff $|k_j| \leq \Lambda$. It is convenient to rewrite Eq. (8) in terms of $r_0 - r_{0c} = a_0 t$ where

$$r_{0c} = -4u_0 n \int_{\mathbf{k}} \mathbf{k}^{-2} \tag{10}$$

is the bulk critical value of r_0 as determined from Eq. (9) (with $\chi_b^{-1} = 0$). Furthermore it is important to separate the $\mathbf{k} = 0$ part $4u_0 n L^{-d} \chi$ from the sum in Eq. (8). After a simple rearrangement we obtain

$$\chi^{-1} = \frac{\delta r_0 + \sqrt{(\delta r_0)^2 + 16 u_0 n L^{-d} (1 + S)}}{2(1 + S)}, \tag{11}$$

where

$$\delta r_0 = a_0 t - \Delta, \tag{12}$$

$$S = 4u_0 n L^{-d} \sum_{\mathbf{k} \neq 0} \left[\mathbf{k}^2 (\chi^{-1} + \mathbf{k}^2)\right]^{-1}, \tag{13}$$

$$\Delta = 4u_0 n \left[\int_{\mathbf{k}} \mathbf{k}^{-2} - L^{-d} \sum_{\mathbf{k} \neq 0} \mathbf{k}^{-2} \right]. \tag{14}$$

These equations are the starting points of our analysis. They are exact in the limit $n \to \infty$ at fixed $u_0 n$ and are valid, at finite cutoff Λ, for $d > 2$, for arbitrary L and for arbitrary r_0. They are written in a form that separates the $\mathbf{k} = 0$ contribution $16 u_0 n L^{-d}$ from the effect of the $\mathbf{k} \neq 0$ modes. The latter is contained in S and Δ.

In addition to the finite-size effect of the $\mathbf{k} = 0$ mode, the $\mathbf{k} \neq 0$ modes cause two different finite-size effects: (i) a finite renormalization of the coupling $u_0 n$ due to S which for $d > 4$ attains the finite bulk value

$$S_b = 4u_0 n \int_{\mathbf{k}} \left[\mathbf{k}^2 (\chi_b^{-1} + \mathbf{k}^2)\right]^{-1}, \tag{15}$$

and (ii) a shift of the temperature scale due to Δ which vanishes in the bulk limit. These two kinds of finite-size effects were also identified by Brézin and Zinn-Justin[3] who argued that for $d > 4$ these effects do not change the leading L dependence obtained within the lowest-mode approximation. These arguments do not depend on n and, if correct, should remain valid also in the large-n limit.

The finite-size effect (ii) comes from Δ which, for $d > 2$ and finite Λ, has the nontrivial large-L behavior

$$\Delta \sim 4u_0 n\Lambda^{d-2}[a_1(d)(\Lambda L)^{-2} + a_2(d)(\Lambda L)^{2-d}], \tag{16}$$

apart from more rapidly vanishing terms. For the coefficients $a_i(d) > 0$ we have found

$$a_1(d) = \frac{d}{3(2\pi)^{d-2}} \int_0^\infty dx x e^{-x} \left[\int_{-1}^1 dy e^{-y^2 x}\right]^{d-1}, \tag{17}$$

$$a_2(d) = \frac{-1}{4\pi^2} \int_0^\infty dy \left[\left(\sum_{m=-\infty}^{\infty} e^{-ym^2}\right)^d - \left(\frac{\pi}{y}\right)^{d/2} - 1\right], \tag{18}$$

as confirmed in Fig. 1 by numerical evaluation of Eq. (14) for $d = 3, 4, 5$. A detailed derivation of Eqs. (16)–(18) is given in Ref. 12. Thus, for $d > 4$, Δ vanishes as L^{-2}, and not as L^{2-d} [3,8–10] or as $L^{-d/2}$.[7]

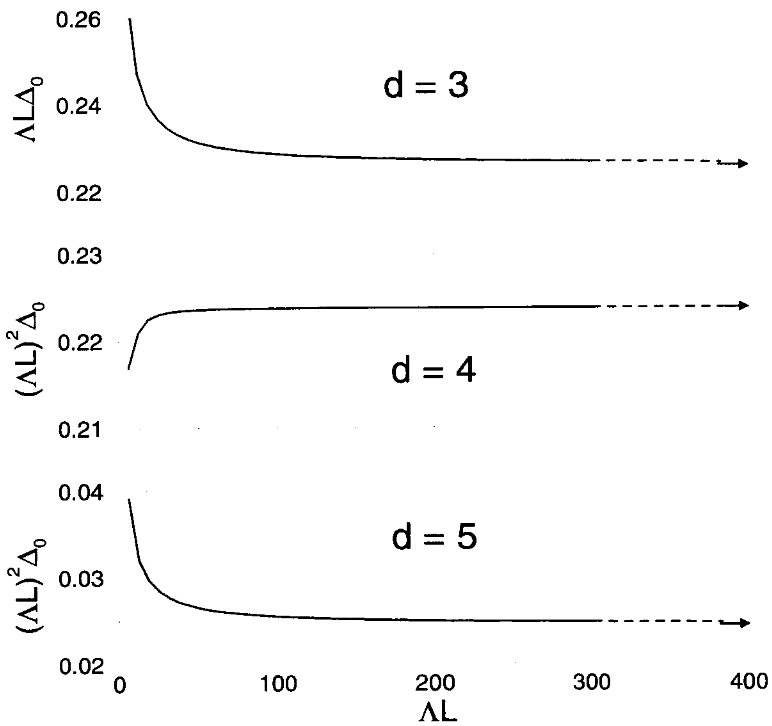

Fig. 1. Numerical calculation of the ΛL-dependence of $\Delta_0 = \Delta/(4u_0 n\Lambda^{d-2})$ with Δ from Eq. (14) for $d = 3, 4, 5$ (solid curves). The dashed lines represent Eq. (16) with $a_1(3) = 0.27706$, $a_1(4) = 0.08333$, $a_1(5) = 0.02443$, and $a_2(3) = 0.22578$, $a_2(4) = 0.14046$, $a_2(5) = 0.10712$, as calculated from Eqs. (17) and (18). The arrows indicate the large-ΛL limits.

The quantity Δ can also be evaluated using the prescriptions of dimensional regularization. This would yield $\Delta = 4u_0 n a_2(d) L^{2-d}$ which agrees with Eq. (16) in the limit $\Lambda \to \infty$ if $2 < d < 4$. For $d \geq 4$, however, the method of dimensional regularization is misleading as it omits the ultraviolet divergence Λ^{d-4} that is tied to the analytic L^{-2} dependence. This is parallel to the omission of a negligible analytic t dependence of ultraviolet-divergent contributions of bulk critical phenomena. In the case of confined systems, however, the analytic dependence on L^{-2} constitutes an important and non-negligible finite-size effect. Here this effect implies that in Eq. (11) the zero-mode term proportional to L^{-d} no longer constitutes the dominant finite-size term for $d > 4$.

Our claims are most convincingly examined at bulk T_c. Then Eq. (11) is reduced to

$$\chi_c^{-1} = \frac{-\Delta + \sqrt{\Delta^2 + 16 u_0 n L^{-d}(1 + S_c)}}{2(1 + S_c)}, \tag{19}$$

where S_c is given by the r.h.s. of Eq. (13) with χ^{-1} replaced by χ_c^{-1}. We see that the large-L behavior is significantly affected by the Δ^2 term. For large L and $d > 4$ we obtain from Eqs. (19) and (16)

$$\chi_c \sim \frac{L^d \Delta}{4 u_0 n} \sim a_1(d) \Lambda^{d-4} L^{d-2}. \tag{20}$$

By contrast, the lowest-mode approximation with $\Delta = 0$ and $S_c = 0$ corresponding to Eq. (5) with $p = 1$ yields the incorrect result

$$\chi_{0c} = (4 u_0 n)^{-1/2} L^{d/2}. \tag{21}$$

We note that the arguments in Refs. 1 and 3 regarding the finite-size effect (ii) are not compelling since they are focused on the contributions of individual terms at lowest nonzero $\mathbf{k}$ rather than on an analysis of the *summed effect* of these contributions. Furthermore we see that the L^{d-2} power law in Eq. (20) differs from the $L^{d/2}$ power law of the exact solution of the n-vector lattice model for $n \to \infty$[13] and of the mean spherical model on a lattice.[14] This proves that the ϕ^4 field theory, Eq. (1), does not correctly describe the leading finite-size effects of spin models with periodic boundary conditions on a d-dimensional lattice with $d > 4$, at least in the large-n limit.

In the following we show that this defect is due to the $(\nabla\varphi)^2$ or $\mathbf{k}^2 \varphi_{\mathbf{k}} \varphi_{-\mathbf{k}}$ term of

$$H = L^{-d} \sum_{\mathbf{k}} \frac{1}{2}(r_0 + \mathbf{k}^2) \varphi_{\mathbf{k}} \varphi_{-\mathbf{k}} + u_0 L^{-3d} \sum_{\mathbf{k}\mathbf{k}'\mathbf{k}''} (\varphi_{\mathbf{k}} \varphi_{\mathbf{k}'})(\varphi_{\mathbf{k}''} \varphi_{-\mathbf{k}-\mathbf{k}'-\mathbf{k}''}) \tag{22}$$

with

$$\varphi_{\mathbf{k}} = \int_V d^d x e^{-i\mathbf{k}\cdot\mathbf{x}} \varphi(\mathbf{x}). \tag{23}$$

Instead we consider a lattice Hamiltonian $\hat{H}(\varphi_i)$ for n-component vectors φ_i with components $\varphi_{i\alpha}$, $-\infty \leq \varphi_{i\alpha} \leq \infty$, $\alpha = 1, \ldots, n$, on lattice points $\mathbf{x}_i$ of a simple cubic lattice with $V = L^d$ and with periodic boundary conditions,

$$\hat{H}/\tilde{a}^d = \sum_i \left[\frac{\hat{r}_0}{2}\varphi_i^2 + \hat{u}_0(\varphi_i^2)^2\right] + \sum_{i,j} \frac{J_{ij}}{2\tilde{a}^2}(\varphi_i - \varphi_j)^2, \tag{24}$$

where J_{ij} is a pair interaction and $\tilde{a}$ is the lattice spacing. In terms of

$$\hat{\varphi}_{\mathbf{k}} = \hat{a}^d \sum_j e^{-i\mathbf{k}\cdot\mathbf{x}_j}\varphi_j, \tag{25}$$

the Hamiltonian $\hat{H}$ has the same form as Eq. (22) but with $r_0 + \mathbf{k}^2$ replaced by $\hat{r}_0 + 2\delta J(\mathbf{k})$ where $\delta J(\mathbf{k}) \equiv J(0) - J(\mathbf{k})$ and

$$J(\mathbf{k}) = (\tilde{a}/L)^d \sum_{i,j} J_{ij} e^{-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}\,. \tag{26}$$

The $\mathbf{k}$ values are restricted by $-\pi/\tilde{a} \leq k_j < \pi/\tilde{a}$. In the large-$n$ limit the susceptibility

$$\hat{\chi} = \frac{1}{n}(\tilde{a}^{2d}/L^d)\sum_{i,j}\langle\varphi_i\varphi_j\rangle \tag{27}$$

is determined by Eqs. (11)–(14) with $\mathbf{k}^2$ replaced by $2\delta J(\mathbf{k})$. The large-L behavior of the crucial quantity $\hat{\Delta}$ is for $d > 2$

$$\hat{\Delta} = 2\hat{u}_0 n \left(\int_{\mathbf{k}} [\delta J(\mathbf{k})]^{-1} - L^{-d}\sum_{\mathbf{k}\neq 0}[\delta J(\mathbf{k})]^{-1}\right) \tag{28}$$

$$\sim 4\hat{u}_0 n J_0^{-1} a_2(d) L^{2-d}, \tag{29}$$

$$J_0 = \frac{1}{d}(\tilde{a}/L)^d \sum_{i,j}(J_{ij}/\tilde{a}^2)(\mathbf{x}_i - \mathbf{x}_j)^2 \tag{30}$$

which for $d > 4$ differs from that of the continuum version Δ, Eq. (16), where $2\delta J(\mathbf{k})$ was approximated by $\mathbf{k}^2$. As a consequence of Eq. (29), the leading L dependence of $\hat{\chi}$ at T_c is for $d > 4$

$$\hat{\chi}_c \sim \frac{1}{2}(\hat{u}_0 n)^{-1/2}(1 + \hat{S}_c^b)^{1/2} L^{d/2} \tag{31}$$

with

$$\hat{S}_c^b = \hat{u}_0 n \int_{\mathbf{k}} [\delta J(\mathbf{k})]^{-2} \tag{32}$$

which differs from Eq. (20) but which agrees with the $L^{d/2}$ power law of the exact solution of the lattice models of Refs. 13 and 14 for $d > 4$ and of the lowest-mode result

$$\hat{\chi}_{0c} = \frac{1}{2}(\hat{u}_0 n)^{-1/2} L^{d/2}\,. \tag{33}$$

An analysis of the temperature dependence of $\chi(t,L)$, Eq. (11), and of $\hat{\chi}(t,L)$ for $d > 4$ yields the asymptotic scaling structure (with $\gamma = 1$ and $\nu = 1/2$)

$$\chi(t,L) = L^{\gamma/\nu} P_\chi(t(L/\xi_0)^{1/\nu}, (L/l_0)^{4-d}), \tag{34}$$

$$\hat{\chi}(t,L) = L^{\gamma/\nu} \hat{P}_\chi(t(L/\hat{\xi}_0)^{1/\nu}, (L/\hat{l}_0)^{4-d}), \tag{35}$$

where ξ_0 and $\hat{\xi}_0$ are the bulk correlation-length amplitudes (above T_c and at vanishing external field) and where

$$l_0 = [4u_0 n(1 + S_c^b)^{-1}]^{1/(d-4)} \tag{36}$$

and

$$\hat{l}_0 = [4\hat{u}_0 n J_0^{-2}(1 + \hat{S}_c^b)^{-1}]^{1/(d-4)} \tag{37}$$

are additional reference lengths. In Ref. 15 it is shown that these reference lengths l_0 and $\hat{l}_0$ are identical with the amplitudes of the bulk correlation length at T_c in the presence of a small external field. The *two-variable* finite-size scaling functions read

$$P_\chi(x,y) = 2\left\{\delta(x) + \sqrt{[\delta(x)]^2 + 4y}\right\}^{-1}, \tag{38}$$

$$\hat{P}_\chi(\hat{x},\hat{y}) = 2J_0^{-1}\left\{\hat{\delta}(\hat{x},\hat{y}) + \sqrt{[\hat{\delta}(\hat{x},\hat{y})]^2 + 4\hat{y}}\right\}^{-1}, \tag{39}$$

where

$$\delta(x) = x - (1 + S_c^b)^{-1} 4u_0 \Lambda^{d-4} n a_1(d) \tag{40}$$

and

$$\hat{\delta}(\hat{x},\hat{y}) = \hat{x} - I_1(J_0^{-1}\hat{P}_\chi^{-1})\hat{y}\,, \tag{41}$$

with I_1 given in Eq. (63) below. Thus $\hat{P}_\chi$ and $\hat{\delta}$ are determined implicitly by Eqs. (39) and (41). An analysis of Eqs. (34)–(39) for fixed $t < 0$ and large L[12] reveals the "dangerous irrelevant" character of the coupling u_0 and $\hat{u}_0$[16] that enters through l_0 and $\hat{l}_0$, in accord with the structure proposed by Privman and Fisher.[17] As an unexpected result, however, P_χ and $\hat{P}_\chi$ differ significantly from each other through the *non-universal* (cutoff dependent and u_0 dependent) shift of the temperature variable x in $\delta(x)$. The lowest-mode form, Eq. (5), would correspond to the *universal* form $\delta(x) = x$ and $\hat{\delta}(\hat{x},\hat{y}) = \hat{x}$ which would imply that both P_χ and $\hat{P}_\chi$ could be rewritten in the *single-variable* form of Eq. (5) with $p = 1$. Because of the non-vanishing shifts in δ and $\hat{\delta}$, however, this is not possible for P_χ (Eq. (38)) and $\hat{P}_\chi$ (Eq. (39)). Thus the structure of Eq. (5) is incorrect for both the continuum and lattice ϕ^4 model.

In the following we extend our analysis to the case $n = 1$ which is of relevance to the interpretation of MC data of the $d = 5$ Ising model.[7–10] First we shall examine χ and the Binder cumulant

$$U = 1 - \langle\Phi^4\rangle/(3\langle\Phi^2\rangle^2) \tag{42}$$

within the field-theoretic φ^4 model, Eq. (1), including the effect of the $\mathbf{k} \neq 0$ modes in one-loop order. For $d > 4$ at finite cut-off, the perturbative finite-size field theory[5,6] is applicable without a renormalization-group treatment. The averages are defined as

$$\langle \Phi^m \rangle = \int_{-\infty}^{\infty} d\Phi \Phi^m P(\Phi), \tag{43}$$

where

$$P(\Phi) = Z^{-1} \int \mathcal{D}\sigma e^{-H} \tag{44}$$

is the order-parameter distribution function with $\sigma(\mathbf{x}) = \varphi(\mathbf{x}) - \Phi$.[6] From Refs. 5 and 6 we derive the L dependence at T_c in one-loop order

$$\chi_c = L^{d/2} u_0^{\mathrm{eff}\,-1/2} \vartheta_2(y_0^{\mathrm{eff}}), \tag{45}$$

$$U_c = 1 - \frac{1}{3}\vartheta_4(y_0^{\mathrm{eff}})/\vartheta_2(y_0^{\mathrm{eff}})^2, \tag{46}$$

where

$$y_0^{\mathrm{eff}} = r_0^{\mathrm{eff}} L^{d/2} u_0^{\mathrm{eff}\,-1/2}, \tag{47}$$

$$r_0^{\mathrm{eff}} = r_{0c} + 12u_0 S_1(r_{0L}) + 144u_0^2 M_0^2 S_2(r_{0L}), \tag{48}$$

$$u_0^{\mathrm{eff}} = u_0 - 36u_0^2 S_2(r_{0L}), \tag{49}$$

$$r_{0L} = r_{0c} + 12u_0 M_0^2, \tag{50}$$

$$M_0^2 = (L^d u_0)^{-1/2} \vartheta_2(r_{0c} L^{d/2} u_0^{-1/2}) \tag{51}$$

with

$$\vartheta_m(y) = \frac{\int_0^{\infty} ds s^m \exp\left(-\frac{1}{2}ys^2 - s^4\right)}{\int_0^{\infty} ds \exp\left(-\frac{1}{2}ys^2 - s^4\right)}. \tag{52}$$

In this order the critical value $r_{0c} < 0$ is determined implicitly by the bulk limit ($r_0^{\mathrm{eff}} = 0$) of Eq. (48),

$$r_{0c} = -12u_0 \int_{\mathbf{k}} (-2r_{0c} + \mathbf{k}^2)^{-1} + 36u_0 r_{0c} \int_{\mathbf{k}} (-2r_{0c} + \mathbf{k}^2)^{-2}. \tag{53}$$

The finite-size effect of the $\mathbf{k} \neq 0$ modes enters through

$$S_m(r) = L^{-d} \sum_{\mathbf{k}\neq 0} (r + \mathbf{k}^2)^{-m}. \tag{54}$$

For large L we have $r_{0L} = -2r_{0c} + \mathcal{O}(L^{-d})$ and

$$\begin{aligned} r_0^{\mathrm{eff}} = &- 12u_0 \left[\int_{\mathbf{k}} (-2r_{0c} + \mathbf{k}^2)^{-1} - S_1(-2r_{0c})\right] \\ &+ 36u_0 r_{0c} \left[\int_{\mathbf{k}} (-2r_{0c} + \mathbf{k}^2)^{-2} - S_2(-2r_{0c})\right] + \mathcal{O}(L^{-d}). \end{aligned} \tag{55}$$

Similar to Δ in Eqs. (14) and (16), the parameter $r_0^{\rm eff} < 0$ vanishes as L^{-2} (rather than as L^{2-d}) for $d > 4$, thus $y_0^{\rm eff} < 0$ *diverges* as $L^{(d-4)/2}$ (rather than *vanishes* as $L^{(4-d)/2}$) for $d > 4$. Since $\vartheta_2(y) \sim -y/4$ and $\vartheta_4(y) \sim y^2/16$ for large negative y[5] this implies that χ_c diverges as L^{d-2} and U_c attains the large-L limit

$$\lim_{L\to\infty} U_c = 2/3. \tag{56}$$

We conclude that the $L^{d/2}$ power law for χ_{0c} and the value

$$U_{0c} = 1 - \frac{1}{3}\vartheta_4(0)/\vartheta_2(0)^2 = 0.2705 \tag{57}$$

predicted[3] for $n = 1$ within the lowest-mode approximation are incorrect for the field-theoretic model. From Refs. 3 and 6 we infer that analogous conclusions hold for general $n > 1$.

These unexpected results show that the widely accepted arguments in support of the asymptotic correctness of the lowest-mode approximation above the upper critical dimension in statics[1,3,4,7–10,14,18] and dynamics[1,19] are not valid and that recent interpretations[7–10] of the MC data of the $d = 5$ Ising model in terms of predictions based on H, Eq. (1), are not conclusive, in spite of the apparent agreement found in Refs. 8–10.

Here we proceed to a new calculation of the Binder cumulant

$$\hat{U} = 1 - \langle\hat{\Phi}^4\rangle/(3\langle\hat{\Phi}^2\rangle^2), \tag{58}$$

$$\hat{\Phi} = L^{-d}\hat{\varphi}_0 = (\tilde{a}/L)^d \sum_j \varphi_j \tag{59}$$

on the basis of the lattice Hamiltonian, Eq. (24), for $d > 4$ and $n = 1$. For large L and small $|t|$ the analytical form of $\hat{U}$ reads in one-loop order

$$\hat{U} = \hat{U}(\hat{x}, \hat{y}) = 1 - \frac{1}{3}\frac{\vartheta_4\left(Y(\hat{x},\hat{y})\right)}{\vartheta_2\left(Y(\hat{x},\hat{y})\right)^2}, \tag{60}$$

$$\hat{x} = t(L/\hat{\xi}_0)^{1/\nu}, \qquad \hat{y} = (L/\hat{l}_0)^{4-d}, \tag{61}$$

$$Y(\hat{x},\hat{y}) = \left[\hat{x} - 12I_1(\bar{r})\hat{y} - 144\vartheta_2(y_0)I_2(\bar{r})\hat{y}^{3/2}\right]\left[\hat{y} + 36I_2(\bar{r})\hat{y}^2\right]^{-1/2}, \tag{62}$$

where

$$I_m(x) = \frac{1}{(2\pi)^{2m}}\int_0^\infty dy y^{m-1} e^{-(xy/4\pi^2)}\left[\left(\frac{\pi}{y}\right)^{d/2} - \left(\sum_{m=-\infty}^{\infty} e^{-ym^2}\right)^d + 1\right], \tag{63}$$

$$\bar{r} = \hat{x} + 12\vartheta_2(y_0)\hat{y}^{1/2}, \qquad y_0 = \hat{x}\hat{y}^{-1/2}. \tag{64}$$

Apart from the bulk length scales $\hat{\xi}_0$ and $\hat{l}_0$ there are no adjustable parameters. The function $\hat{U}(\hat{x},\hat{y})$ is illustrated in Figs. 2–4 for $d = 5$. At one-loop order, the asymptotic ($T = T_c, L \to \infty$) value of $\hat{U}$ coincides with the value given in Eq. (57). It is suggestive to expect that the lattice model, Eq. (24), for $n = 1$ exhibits the same

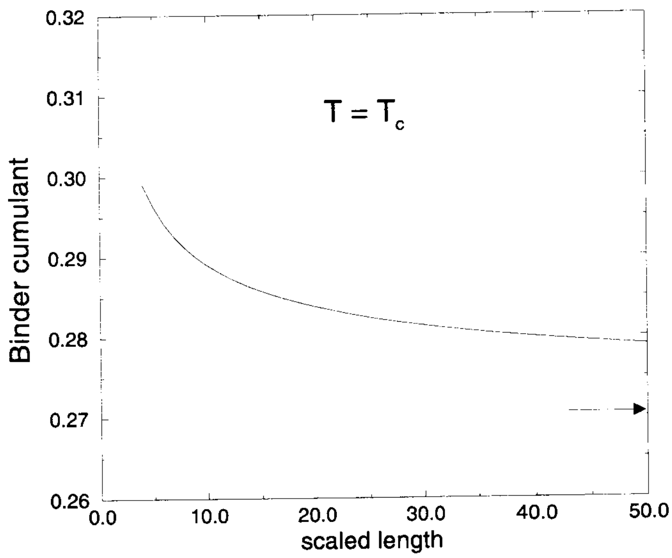

Fig. 2. Theoretical prediction of the Binder cumulant $\hat{U}(0,\hat{y})$, Eq. (60), at T_c and $d = 5$ as a function of the scaled length $\hat{y}^{-1} = L/\hat{l}_0$ from Eq. (61). The arrow indicates the asymptotic one-loop value $\hat{U}(0,0) = 0.2705$ for $L \to \infty$.

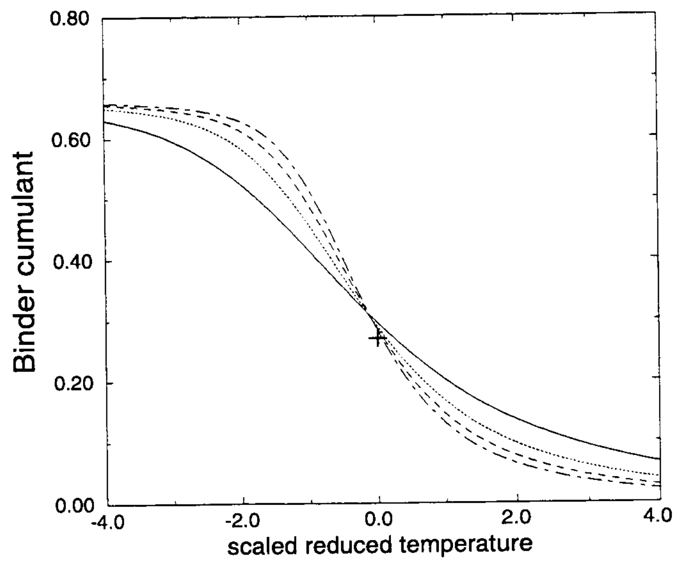

Fig. 3. Theoretical prediction of the Binder cumulant $\hat{U}(\hat{x},\hat{y})$, Eq. (60), as a function of the scaled reduced temperature $\hat{x} = t(L/\hat{\xi}_0)^{1/\nu}$, Eq. (61), for several values of the scaled length $\hat{y}^{-1} = (L/\hat{l}_0)^{d-4}$ from Eq. (61), at $d = 5$: $L/\hat{l}_0 = 5$ (solid line), $L/\hat{l}_0 = 10$ (dotted line), $L/\hat{l}_0 = 15$ (dashed line), $L/\hat{l}_0 = 20$ (dot-dashed line). The cross + indicates the asymptotic one-loop value $\hat{U}(0,0) = 0.2705$ at $T_c, L \to \infty$.

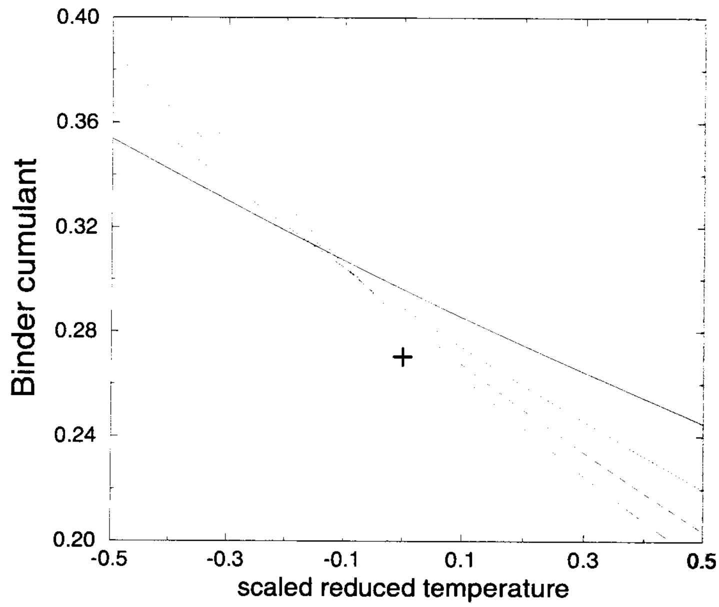

Fig. 4. Same plot as in Fig. 3 with enlarged scales. The different lines cross each other in a finite range around $\hat{x} = -0.1$. The MC value of Rickwardt *et al.*[7] is $g_L(T_c) = -0.958 \pm 0.050$ corresponding to $\hat{U} = -g_L/3 = 0.319 \pm 0.017$, which is in the range within which our lines cross each other. The cross + indicates the asymptotic one-loop value $\hat{U}(0,0) = 0.2705$ at $T_c, L \to \infty$.

asymptotic finite-size effects as Ising-like systems. Therefore it would be interesting to reanalyze previous MC data for the $d = 5$ Ising model on the basis of our new results. The derivation of Eqs. (60)–(64) will be given elsewhere.

On the basis of our exact results in the large-n limit and our one-loop results for $n = 1$ we summarize our main findings (for cubic geometry and periodic boundary conditions and for $d > 4$) as follows. (i) The ϕ^4 *field theory* yields a *nonuniversal* finite-size scaling function for χ with a temperature shift $\sim L^{-2}$ and a power law L^{d-2} at T_c. (ii) The ϕ^4 *lattice model* yields a finite-size scaling function for $\hat{\chi}$ that differs significantly from that of the field theory; it yields a power law $L^{d/2}$ at T_c and a complicated (implicitly determined) shift of the temperature variable, Eq. (41). (iii) The lowest-mode approach[3] yields *universal* finite-size scaling functions for both the field-theoretic and the lattice ϕ^4 model, in disagreement with our results. The lowest-mode approach happens to yield the correct power law $\hat{\chi}_c \sim L^{d/2}$ at T_c for the lattice model, but not for the continuum model. (iv) The universal result for the Binder cumulant predicted by the lowest-mode approach[3] cannot be justified on the basis of the ϕ^4 *continuum* theory. Our one-loop result for the asymptotic value of the Binder cumulant $\hat{U}_c$ of the *lattice* model for $n = 1$ coincides with the lowest-mode result $\hat{U}_{0c} = U_{0c}$, Eq. (57).

We also anticipate lattice and cutoff effects on leading finite-size terms at $d = d_u$ for periodic boundary conditions. This is relevant to future MC studies of tricritical phenomena at $d = 3$, as well as to MC simulations for four-dimensional Ising models[20,21] and for lattice models of elementary particle physics at $d = 4$.

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