FESTKÖRPER PROBLEME

ADVANCES IN SOLID STATE PHYSICS

tents

Critical Dynamics near the λ-Transition in ^{4}He

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Summary: Universality and dynamic scaling behavior are discussed in the context of the critical dynamics near the lambda transition in ^{4}He. By means of a field-theoretic renormalization-group approach the entire region between asymptotic criticality and noncritical background behavior is studied. This approach provides a quantitative description of the observable dynamics in terms of nonasymptotic crossover functions rather than in terms of asymptotic power laws. It identifies two reasons for the experimentally observed large departures from universality and dynamic scaling: i) the possible instability of the dynamic-scaling fixed point, ii) a strong decrease of the effective mode-coupling constant towards a small background value. This resolves longstanding problems in the critical dynamics above and below the lambda transition and opens up the possibility of further detailed tests of the renormalization-group theory.

1 The Problems

Universality and scaling behavior are central characteristics in the modern renormalization-group theory of critical phenomena, both in statics and dynamics. Sufficiently close to the critical temperature, at sufficiently long wave lengths and low frequencies, the theory predicts the existence of an asymptotic critical region where correlation functions depend on temperature, wave vector and frequency in a universal way, independent of the specific nature of the microscopic (short-range) interaction. Only the dimensionality d of the system, the number n of components of the order parameter and - in dynamics - certain conservation laws determine the universal quantities such as critical exponents, scaling functions and amplitude ratios. In a region farther away from criticality, the leading power laws are smoothly modified by subleading correction terms, again described by power laws with universal transient exponents and nonuniversal amplitudes. This description in terms of power laws seemed to be adequate, within the accuracy of the available experiments, for the explanation of dynamic critical phenomena in various systems [1, 2].

Striking discrepancies, however, existed with the observed dynamics near the lambda transition in ^{4}He [1]. This was particularly disturbing for theorists and experimentalists because this transition was considered to be most favorable for a conclusive test of the theory, as demonstrated successfully in the case of static critical phenomena [3]. Remarkable experimental advantages are the absence of strains and inhomogeneities, the smallness of gravitational effects, as well as the existence of a *line* of critical points, the λ-line $T_\lambda(P)$. This offers the possibility of a direct experimental test of unversality by studying the pressure P dependence. Various experiments, however, disagreed with the detailed quantitative predictions of dynamic scaling, mode-coupling and renormalization-group theories [4–8]. Specifically, the problems were the following:

(i) Thermal conductivity measurements [9] very close to T_λ, in the range

$$10^{-5.5} \leqslant t \leqslant 10^{-3}, \quad t = \frac{T - T_\lambda(P)}{T_\lambda(P)}, \tag{1}$$

yielded an effective critical exponent which was about 20 % larger than predicted by the theory [4–6]. This was surprising because the range (1) was expected to be well within the asymptotic critical region (apart from a smooth modulation due to the almost logarithmic temperature dependence of the specific heat).

(ii) A related difficulty was the observed nonuniversal pressure dependence (by (by about 30 % at $t = 10^{-5}$) of the dimensionless amplitude R_λ [6] entering the thermal conductivity [9], contrary to the expected universality [6] of R_λ. This experimental fact clearly demonstrated that the universal asymptotic region must be much smaller than $t = 10^{-5}$, for reasons of unknown origin.

(iii) The weak temperature dependence of the linewidth Γ_2 of the measured light-scattering spectrum [10–13] above and below T_λ was in gross contradiction with the power law

$$\Gamma_2 \sim \xi^{1/2}, \xi \sim |t|^{-\nu}, \nu \approx 2/3, \tag{2}$$

predicted by dynamic scaling theory in the hydrodynamic region ($k\xi < 1$) [4–8].

(iv) The ratio of the transport coefficients Γ_ξ and Γ_κ entering the hydrodynamic form of the dynamic structure factor below T_λ was measured [10–13] to be of the order of 2, in contrast to the theoretical value 0.36 [14].

(v) Another striking discrepancy between theory [7, 8] and experiment [15] for the amplitude of second-sound damping was recently reduced by new experiments [16]. Nevertheless no reliable theoretical prediction for this amplitude was available.

(vi) A serious problem of purely theoretical nature was the poor convergence of the ϵ (=4 − d) expansion employed in the asymptotic renormalization-group theory [1, 6, 7]. For example, the asymptotic ratio w^* between order parameter and the entropy relaxation rate was found to be $w^* = 1 - 1.41\,\epsilon$ [6]. This could not be properly extrapolated to $\epsilon = 1$ to yield a reliable (non-negative) value in three dimensions for w^* and for the universal amplitudes and scaling functions depending on w^*.

In the present review we shall describe the main points in the recent theoretical developments which have essentially resolved all of these problems. It will become clear why the previous asymptotic power-law descriptions fail to explain the observable ^{4}He dynamics, and that there is now satisfactory agreement between a nonasymptotic renormalization-group approach [17–20] and the previous [9–13] and new [16, 21, 22] experiments. For related recent reviews on this topic see also [3, 23, 24].

2 Asymptotic Theory: Universality and Dynamic Scaling

The asymptotic critical dynamics of the lambda transition can be described by the well-known model E of Halperin, Hohenberg, and Siggia [6]. It consists of coupled Langevin-equations for the complex order parameter $\psi(\underline{x}, t)$ and the entropy density $m(\underline{x}, t)$

$$\dot{\psi} = -2\Gamma_0 \frac{\delta H}{\delta \psi^*} - i g_0 \psi \frac{\delta H}{\delta m} + \Theta_\psi, \tag{3}$$

$$\dot{m} = \lambda_0 \nabla^2 \frac{\delta H}{\delta m} + 2 g_0 \,\mathrm{Im}\left(\psi^* \frac{\delta H}{\delta \psi^*}\right) + \Theta_m, \tag{4}$$

$$H = \int d^d x \left\{\frac{1}{2}\tau_0 |\psi|^2 + \frac{1}{2}|\nabla\psi|^2 + u_0 |\psi|^4 + \frac{1}{2}\chi_0^{-1} m^2\right\}, \tag{5}$$

with real kinetic coefficients Γ_0, λ_0, and couplings u_0, g_0, and with Gaussian random forces Θ_ψ, Θ_m. These equations are based on the hydrodynamics of ^{4}He near T_λ [25] and are believed to provide a complete description of the asymptotic critical behavior in three dimensions.

Convenient dynamic paramters are the ratio of relaxation rates

$$w_0 = \frac{\Gamma_0 \chi_0}{\lambda_0} \tag{6}$$

and the dimensionless coupling constant

$$f_0 = \frac{g_0^2\, \xi_0}{2\pi^2 \Gamma_0 \lambda_0} \tag{7}$$

where ξ_0 is the amplitude of the correlation length above T_λ. These bare values w_0 and f_0 can be adjusted to the noncritical behavior well away from the superfluid transition.

As a consequence of universality, the critical dynamics is governed by fixed point values w^* and f^* which are independent of the bare ones. They are determined, within the field-theoretic renormalization-group approach [26–28], as the zeros of the β-functions

$$\beta_w(w^*, f^*) = 0, \beta_f(w^*, f^*) = 0. \tag{8}$$

These functions can be calculated by means of renormalized perturbation theory. They are presently known in two-loop order, i.e. second order in

$$f^* \sim 0 \ (\epsilon = 4 - d) \tag{9}$$

and to all orders in w^* [28, 29]. (The dependence on the small static coupling $u^* \sim \epsilon/40 \ll f^*$ is unimportant in three dimensions.)

According to the physical meaning of w^* as the asymptotic ratio of relaxation rates, the order parameter and the entropy relax on the same time scale if

$$w^* > 0. \tag{10}$$

In this case (extended) dynamic scaling [4, 5] holds, which implies a dynamic critical exponent (both for the order parameter and the entropy correlation function) [4]

$$z = 3/2 \tag{11}$$

in three dimensions. This exponent enters the asymptotic scaling expression for the thermal conductivity [6]

$$\lambda_{th} = g_0 \chi_0^{1/2} R_\lambda \xi^{2-z} \tag{12}$$

whose amplitude [6, 28, 30]

$$R_\lambda = (2\pi^2 w^* f^*)^{-1/2} [1 - \frac{f^*}{4} + 0(f^{*2})] \tag{13}$$

is a universal quantity [6]. By contrast, the experiments showed a significant nonuniversal dependence of this amplitude on t and on the pressure. [9]

A possible breakdown of the traditional dynamic-scaling picture in three dimensions was substantiated theoretically in a study by De Dominicis and Peliti [28] of an n-dependent generalization of model E [31, 32]. These authors discovered the possible physical relevance of a boundary in the n-d plane (Fig. 1) which separates a scaling region, where $w^* > 0$ is the stable fixed point, from a weak-scaling region where $w^* = 0$ becomes the stable fixed point. In the latter case extended dynamic scaling breaks down and, instead of (11), two different exponents z_ψ and z_m govern the asymptotic critical dynamics.

The stability boundary emerges from d = 4, n = 3/2 [32, 33] with a linear extrapolation [28, 34] *to the vicinity of the physical point* n = 2, d = 3 (full and dashed line in Fig. 1). The small distance of the physical point from the boundary has two important consequences for the ^{4}He dynamics in three dimensions:

(I) a small dynamic transient exponent $\omega_W \ll 1$ and therefore a small asymptotic scaling region [28],

(II) a small asymptotic ratio $w^* \ll 1$ [29, 30, 35].

In particular, the closeness of the boundary to the point n = 2, d = 3 implies $w^* \ll \epsilon = 4 - d$ and therefore explaines [30] why ϵ is not the appropriate smallness parameter at n = 2 near d = 3. Instead, extrapolation procedures which properly include the vanishing of w^* and ω_W along the boundary [29, 30, 35] avoid the difficulty [6, 7] of a direct extrapolation of the ϵ-expansion results to $\epsilon = 1$ [problem (vi) in Section 1]. Alternatively, one can solve directly for the zeros w^*, f^* of the (n = 2) β-functions in Eq. (8) [36] which yields $w^* = 0.02$ and $f^* = 0.83$ in three dimensions, apart from higher-loop corrections.

The sign and magnitude of the *curvature* of the boundary near d = 4 is at present unknown. Our detailed analysis [20] of the most recent thermal conductivity data [21] suggests that the true (curved) boundary lies somewhere in the shaded region in Fig. 1. This analysis does not, however, allow conclusions about the global shape of the boundary in the n–d plane for $d \lesssim 3$.

Very recently, new information about this problem has been obtained [37] by means of a study of the asymptotic critical dynamics in $2 + \epsilon$ dimensions. It was found that $w^* = 0$, $f^* = \infty$ becomes the stable fixed point near d = 2 for all n > 2. This is consistent with a boundary having an upward curvature which may put the physical point n = 2, d = 3 in the weak-scaling region. Furthermore, an additional boundary should exist in the n-d plane where the fixed point with $f^* = \infty$ becomes stable [18, 37].

The small w^* has significant effects on the asymptotic shape and temperature dependence of the light-scattering spectrum [30, 35, 38], on the magnitude of second-sound damping [39], and on the thermal conductivity [see equation (13)]. These effects are most pronounced in the asymptotic critical region which, however, is not experimentally accessible. In particular, the effect on the temperature dependence of the halfwidth Γ_2 which results from the small w^* [38] is relevant in this asymptotic region but does not explain the observed temperature dependence

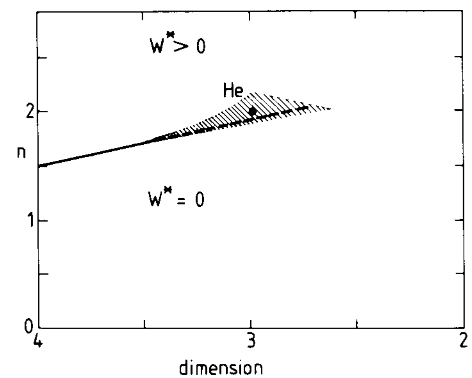

Fig. 1
Linear extrapolation (full and dashed line) [28, 34] for the stability boundary of the SSS-model [31, 32]. Our analysis of recent experimental data suggests that the true (curved) boundary lies somewhere in the shaded region. From Ref. [20].

of Γ_2 at finite wave number, as was first recognized by Ferrell and Bhattacharjee [40–42] (see also [43]). Therefore the discrepancies with the experiments mentioned in Section 1 require an explanation which is based on a nonasymptotic theory.

3 Nonasymptotic Theory: Crossover Behavior

A quantitative analysis of the observable dynamics above T_λ in terms of corrections to scaling was first carried out by Ferrell and Bhattacharjee [40–42]. The corrections were taken to be of the usual power-law type, with the small transient exponent $\omega_w \ll 1$ and adjustable nonuniversal amplitudes. These authors also opened up a more general point of view by incorporating in their analysis the "precritical" behavior far from criticality, which they described by means of a "high-temperature" expansion in powers of the correlation length [40–42, 44]. These two approaches yielded results which were in good agreement with the thermal conductivity and light-scattering data above T_λ. On the other hand, these descriptions were not satisfactory in the experimental region because they were based on asymptotic power laws whose actual validity is limited to the (inaccessible) vicinity of the λ point and to the precritical region well away from T_λ.

A complete theory which treats the entire region between criticality and noncritical background behavior above and below T_λ was introduced by the present authors [17, 18] in terms of a nonlinear renormalization-group analysis. This approach includes both the limiting power-law behaviors close to and far from T_λ as well as the complicated *crossover between these limits* in the experimentally accessible regime. Such an approach is made possible by means of an appropriate application of the field-theoretic renormalization-group [26–28]. Independent suggestions somewhat similar in spirit were also made by other authors [36, 45] although at that time they did not recognize the necessity of a fully nonlinear treatment.

We first sketch our nonlinear analysis for the example of the thermal conductivity λ_{th} since it is quite sensitive to the small w^* and to the small transient exponent $\omega_w \sim w^*$ which strongly affects the observable dynamics, as we shall see.

Within the field-theoretic renormalization-group approach the departures from the asymptotic form of λ_{th}, Eqs. (12), (13), can be described in a compact way by generalizing the amplitude R_λ to an *effective amplitude* [17, 18, 36]

$$R_\lambda^{eff} = [2\pi^2\, w(\ell)\, f(\ell)]^{-1/2}[1 - f(\ell)/4]. \qquad (14)$$

Here $w(\ell)$ and $f(\ell)$ are effective parameters determined by the renormalization-group flow equations

$$\ell \frac{dw}{d\ell} = \beta_w(w, f), \quad \ell \frac{df}{d\ell} = \beta_f(w, f). \qquad (15)$$

The β-functions, in two-loop order, have the structure [28, 29]

$$\beta_w = A(w)\, f + B(w)\, f^2 + 0(f^3), \qquad (16)$$

$$\beta_f = (d-4)\, f + C(w)\, f^2 + D(w)\, f^3 + 0(f^4). \qquad (17)$$

The asymptotic conditions, Eq. (8), for the fixed point values $w^* = w(0)$ and $f^* = f(0)$ are recovered from Eq. (15) in the limit $\ell \to 0$. In our example, the flow parameter ℓ can be identified as

$$\ell = t^\nu. \qquad (18)$$

In general, outside the hydrodynamic region, ℓ is related also to k and ω via an appropriate matching condition.

Rather than studying the usual asymptotic case $\ell \to 0$, one should exploit the complete information about the nonasymptotic dynamics contained in Eqs. (15)–(17) by integrating these equations. The result can be represented in terms of a flow diagram for the effective parameters $w(\ell)$ and $f(\ell)$ (Fig. 2) [18]. For not too large f, this provides conclusive information in the entire range $0 \leq w \leq \infty$ because the w-dependence in Eqs. (16, 17) is known exactly [28, 29]. Fortunately enough, the ^{4}He dynamics in three dimensions is indeed represented by the *small-f-domain* in Fig. 2 [18], where the expansion with respect to f is reasonable. The large-f-domain may become relevant near two dimensions [18, 36, 37].

Our argument for the necessity of a complete integration of the flow equation is the following. The leading corrections $\sim \ell^{\omega_w}$, as obtained from a linearization of the β-functions, are followed by subleading correction terms $\sim \ell^{2\omega_w}, \ell^{3\omega_w}, \ldots$ (from the nonlinear w-dependence of the β-functions). The latter terms were negligible if $\omega_w \sim 0(1)$. Because of $\omega_w \ll 1$, however, many of these terms are comparable in magnitude with the term $\sim \ell^{\omega_w}$ in an appreciable ℓ-range and thus become nonnegligible. Therefore the standard concept of an expansion of the β-functions around the fixed point becomes inapplicable.

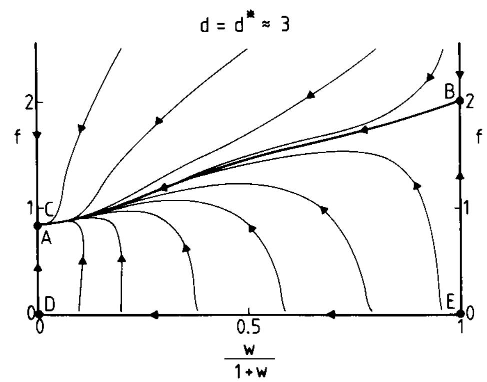

Fig. 2
Flow diagram for the dynamic parameters of model E in three dimensions according to Eqs. (15)–(17). From Ref. [18].

The salient features of the critical dynamics of the lambda transition follow from the flow diagram in Fig. 2. The dynamics at a particular pressure are represented by a particular trajectory. The pressure dependence enters via the nonuniversal initial conditions. They can be identified via a comparision with experimental data (see Section 4), in practice by a fit with adjustable parameters $w(\ell_0)$ and $f(\ell_0)$ at some convenient initial point ℓ_0. For sufficiently large ℓ_0, all trajectories (of the small-f-domain) start from the vicinity of the $f = 0$ axis which represents the background behavior with a small dynamic coupling $f(\ell_0)$. As ℓ decreases, $f(\ell)$ increases rapidly to $O(1)$. From then on, as $\ell \to 0$, all trajectories flow slowly towards a unique fixed point $w^* \approx 0$, $f^* \sim O(1)$, in accord with universality. The asymptotic universal behavior, however, is not observable because of the slowing down of the trajectory flow near the fixed point ($\omega_W \ll 1$), due to the closeness of the weak-scaling region to the physical point $n = 2$, $d = 3$ (Fig. 1).

The high-temperature expansion of Ferrell and Bhattacharjee (in powers of $\ell^{-1} \sim \xi$) [41, 42, 44] applies to the region where $f(\ell)$ is still small. The connection of this expansion with the nonlinear renormalization-group approach was recently discussed by Ahlers, Hohenberg and Kornblit [46].

An appealing feature of the renormalization-group approach is that it easily allows one to make quantitative predictions about the nonasymptotic, i.e. nonuniversal critical dynamics of all other physical quantities above and below T_λ, *without further adjustable parameters*. There is no nonuniversal parameter left once the particular ^{4}He-trajectory in Fig. 2 has been identified (via R_λ^{eff}, for example). We illustrate this point in sections 6 and 7 for the critical dynamics in the superfluid phase, after an identification of the trajectory via R_λ^{eff} in the normal phase.

4 Thermal Conductivity

The thermal conductivity data of Ahlers [9] at saturated vapor pressure are shown in Fig. 3. They provide sufficient information for an approximate identification of the trajectory in the flow diagram shown in Fig. 2. The data for $t < 10^{-4}$ deviate systematically from the dynamic-scaling behavior which is most clearly exhibited by plotting the experimental values for

$$R_\lambda^{exp} = \frac{\lambda_{th}}{g_0\, \xi^{1/2}\, C_p^{1/2}} \tag{19}$$

as a function of t (Fig. 4). They are in definite disagreement with the temperature independent universal value given in Eq. (13). The nonuniversal temperature dependence of R_λ^{exp} is well reproduced by a fit of R_λ^{eff}, Eq. (14), in the range $t < 10^{-3}$, with adjustable initial conditions $w(\ell_0)$ and $f(\ell_0)$ at $\ell_0 = 10^{-2}$ ($t = 10^{-3}$) (full curve in Fig. 4). The corresponding effective parameters $w(\ell)$ and $f(\ell)$ (Fig. 5) identify the ^{4}He trajectory in the small-f-domain of the flow diagram (Fig. 2) and reveal two important crossover effects [17, 18]:

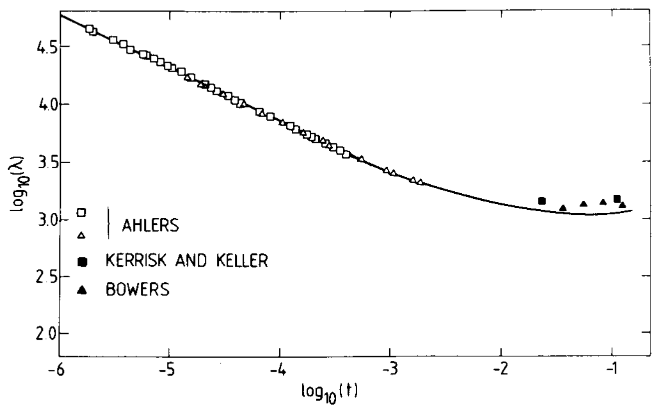

Fig. 3 Comparison between the nonlinear renormalization-group theory (full curve) and experimental data [9, 49] for the thermal conductivity at saturated vapor pressure. From Ref. [18].

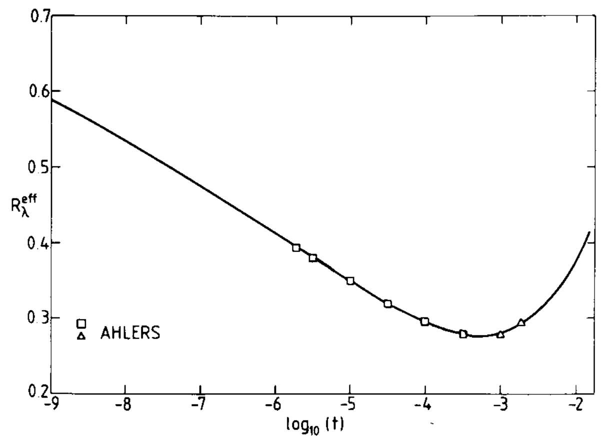

Fig. 4 Comparison between theory (full curve) and representative experimental data [9] for the effective amplitude R_λ^{eff} of the thermal conductivity at saturated vapor pressure. From Ref. [18].

(i) close to T_λ a slow ($\omega_W < 0.1$) decrease of $w(\ell)$ towards a small fixed point value $w^* = w(0)$; this explains the increase of R_λ^{exp} for $t < 10^{-4}$;

(ii) farther away from T_λ a strong decrease of $f(\ell)$ towards a small background value $f(1) \ll 1$; this predicts a strong increase of R_λ^{exp} for $t > 10^{-3}$.

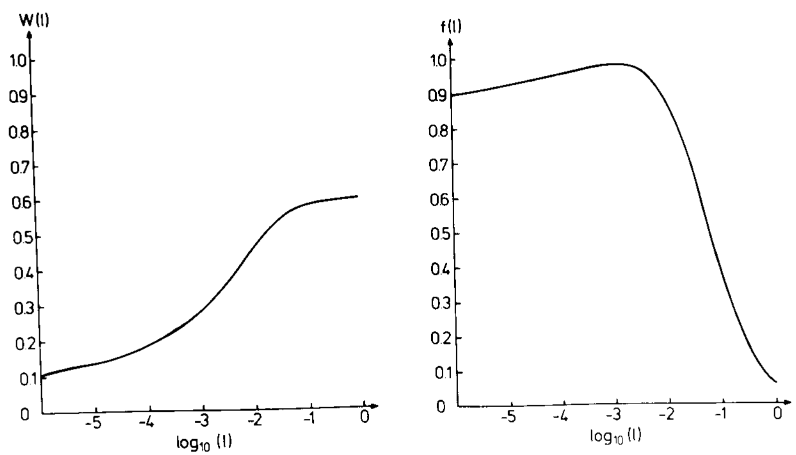

Fig. 5 Effective dynamic parameters of model E corresponding to ^4He at saturated vapor pressure. From Ref. [18].

Both effects are present also at higher pressure [19], with some quantitative differences. In particular, the crossover temperature t_c where the strong decrease of f (ℓ) sets in is shifted towards a smaller value because of the smaller background value f (1) at higher pressure [19]. The connection between the smallness of t_c and of f (1) was pointed out by Ahlers, Hohenberg and Kornblit [47]. Quite remarkable is the fact that w (ℓ) and f (ℓ) exhibit a measurable pressure dependence even for $t \lesssim 10^{-5}$ which explains the 30 % nonuniversal variation of R_λ^{exp} noted in Section 1 (the variation is best demonstrated in Fig. 2 of [46]). This is again an important effect of the small ω_W induced by the weakscaling fixed point. [48]

In order to test the prediction noted in point (ii) above one can integrate the flow equations towards *larger* ℓ and compare the ensuing λ_{th} with experiments well away from T_λ [18]. The resulting semiquantitative agreement of the full curve in Fig. 3 with the background data [49] constitutes a major advance of our global, nonlinear renormalization-group approach *because it does provide a bridge between noncritical background and asymptotic critical behavior, in contrast to all power-law approximations* [17, 18, 36, 40–42]. Some refinements of the theory can now be made to improve its accuracy.

5 Model F Analysis

It was pointed out by Halperin, Hohenberg and Siggia [6] that in the experimentally accessible region, away from criticality, the Hamiltonian (5) has to be complemented by additional terms

$$\gamma_0 \, m |\psi|^2 \qquad (20)$$

and $h_0 m$, and Γ_0 in (3) should be taken to be complex. This is model F [6]. It has the same asymptotic behavior as model E in three dimensions where the fixed point value γ^* of the static coupling (20) vanishes. In the nonasymptotic region, however, the finite (though small) effective coupling γ (ℓ) will cause some additional corrections to the model E results [of perhaps 0 (15 %)] which should be taken into account in a fully quantitative theory.

Such a more refined analysis became of particular interest since recently more accurate thermal conductivity [21] and second-sound damping data [22] have been published. An extensive model F analysis was recently carried out by Ahlers, Hohenberg, and Kornblit [46,47]. Their results yielded further confirmation for the basic features of the parameter flow w (ℓ) and f (ℓ) discussed in Section 4 above. Furthermore, the authors emphasized the smallness of f (ℓ) in the precritical region and the ensuing validity of perturbation theory. On the other hand, their explicit quantitative results suffered from an unsystematic (i.e. not loop-wise) treatment of model F. This implied misleading conclusions about the dynamics close to criticality [20, 23].

A more reliable model F analysis became possible only very recently when the complete two-loop flow equations were calculated by one uf us [50]. As far as the thermal conductivity is concerned, the main results of this analysis are [20, 51]:

(a) Model F in two-loop order is in excellent agreement with the new thermal conductivity data of Ahlers and Behringer [21] at saturated vavor pressure, as demonstrated in Fig. 6.

(b) The analysis predicts a further increase of the amplitude of the thermal conductivity close to T_λ, according to Fig. 6.

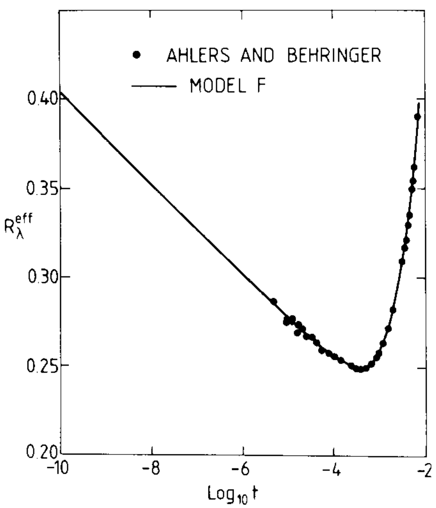

Fig. 6
Comparison between a model F fit (full curve) and new experimental data [21] for the effective amplitude R_λ^{eff} of the thermal conductivity. From Ref. [20].

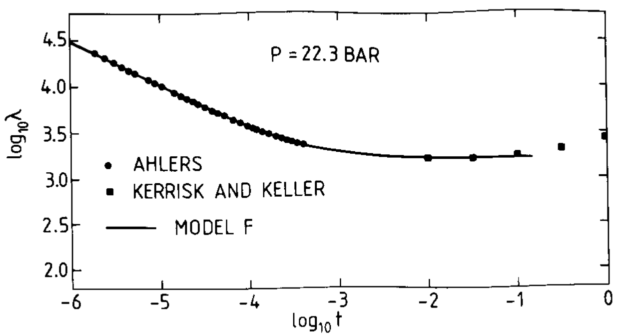

Fig. 7 Comparison between a model F fit (full curve) and experimental data [9, 49] for the thermal conductivity at higher pressure. From Refs. [20, 51].

(c) The present thermal conductivity data suggest that the physical point $n = 2$, $d = 3$ in the n-d plane (Fig. 1) is close to the weak-scaling region.

Excellent agreement of a model F fit is also found [51] with the thermal conductivity data at higher pressure [9, 49] over five decades in relative temperature (Fig. 7), apart from very small deviations for $t < 10^{-5}$ [20].

It should be noted, on the other hand, that further confirmation of these points is necessary, both from the experimental and theoretical side. Specifically, the unexplained cellsize dependence of the data at saturated vapor pressure [46, 52] must be elucidated. Also the curvature of the boundary in the n-d plane (Fig. 1) should be determined near $d = 4$ by means of a three-loop calculation [34].

6 Second-Sound Damping

Once the particular renormalization-group trajectory of the ^{4}He dynamics is identified, one can calculate the entire crossover between precritical and asymptotic critical behavior of any physical quantity *without further adjustable parameters* [17]. Specifically we consider here the damping coefficient D_2 of second sound below T_λ. It is well suited for a test of our theory since the temperature dependence of D_2 is known experimentally [16, 22, 53] between background and criticality over four decades of the relative temperature $|t|$ (Fig. 8).

The asymptotic scaling form of D_2 reads [1]

$$D_2 = 2\, R_2\, c_2\, \xi_- \tag{21}$$

with a universal amplitude [1, 7, 39]

$$R_2 = \frac{A}{(w^* f^*)^{1/2}}\, [1 + w^* + O(f^*)]. \tag{22}$$

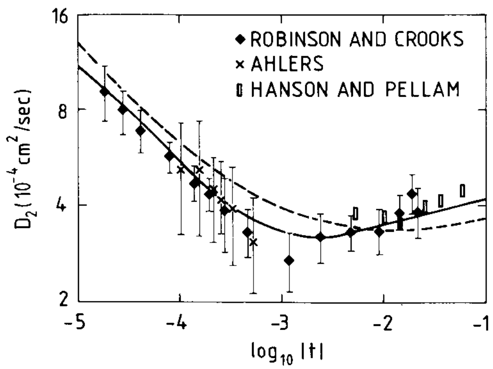

Fig. 8
Comparison between theory (model E: dashed curve, model F: full curve) and experimental data [16, 22, 53] for the damping D_2 of second sound. From Ref. [20].

c_2 and ξ_- are the secound-sound velocity and the correlation length below T_λ, respectively. The constant A in (22) depends on static quantities only. Quite analogous to R_λ above T_λ, the asymptotic universal amplitude R_2 can be generalized to an effective amplitude [17, 19]

$$R_2^{eff} = \frac{A}{[w(\ell)\, f(\ell)]^{1/2}}\, [1 + w(\ell) + O(f(\ell))]. \tag{23}$$

This amplitude is now temperature dependent through $w(\ell)$ and $f(\ell)$ (see Fig. 5) according to the connection between ℓ and t below T_λ,

$$\ell = (-2\,t)^\nu. \tag{24}$$

The leading dependence of R_2^{eff} on $w(\ell)$ and $f(\ell)$ is similar to that of R_λ^{eff}, Eq. (14), except for the additional factor $1 + w(\ell)$ in Eq. (23). This factor makes R_2^{eff} less sensitive with respect to the decreasing $w(\ell)$ for $-t < 10^{-4}$ (Fig. 9) and hence the temperature dependence of D_2 deviates only weakly from dynamic scaling behavior [1, 4, 5] (Fig. 8) for $-t < 10^{-4}$, as predicted [17]. Like R_λ^{eff} above T_λ, the observable R_2^{eff} is quite far from its asymptotic value R_2, as indicated in Fig. 9, which is again due to the slow transient ω_W induced by the weak-scaling fixed point. Also, like R_λ^{eff} above T_λ, it is clear that R_2^{eff} must be pressure dependent via $w(\ell)$ and $f(\ell)$ [46, 54].

Farther from T_λ our theory predicts an increasing R_2^{eff} due to the decreasing $f(\ell)$ as $\ell \to 1$ (Fig. 5, 9). The resulting $D_2 = 2\, R_2^{eff}\, c_2\, \xi_-$ agrees reasonably well with experiment both for models E and F (dashed and full curves in Fig. 8, respectively), with improved accuracy in case of model F [20, 46]. Here c_2 and ξ_- have been taken from experiment [9]. A theory which provides a satisfactory quantitative treatment also of the static factor $A\, c_2\, \xi_-$ of D_2 is still lacking [19]. A corresponding treatment would consist in a complete nonasymptotic study of D_2 within model F below T_λ.

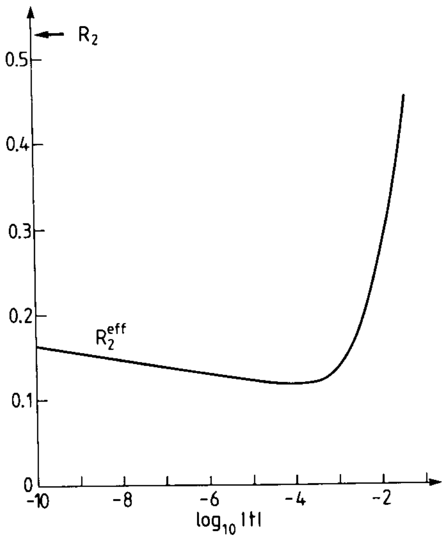

Fig. 9
Effective amplitude, Eq. (23), corresponding to the full curve for D_2 in Fig. 8. The arrow indicates the asymptotic universal amplitude $R_2 = R_2^{eff}(t=0)$.

Recently the second-sound damping D_2 in the precritical region ($|t| > 10^{-2.5}$) was calculated by means of the "high-temperature" expansion [54]. The result is lower than the dashed and full curves in Fig. 8 which seems to be due mainly to a different treatment of the static quantities entering D_2. This disagreement needs further investigation.

7 Light-Scattering Spectrum

A different test of the theory is provided by comparing the entropy correlation function $< mm > (k, \omega)$ with light-scattering experiments which also probe the wavenumber (k) and frequency (ω) dependence of the critical dynamics. The necessity for a nonasymptotic theory of the light-scattering spectrum is due to the relatively large wavenumber $k \sim 10^5$ cm^{-1} of the experiments [10–13], as noted by Ferrell and Bhattacharjee [40]. By means of a semiquantitative theory including corrections to scaling they obtained reasonable agreement [40–42, 55] with the absence of temperature dependence in the (rather inaccurate) data for the linewidth above T_λ [12].

A more complete description for $T \geq T_\lambda$ follows from generalizing the asymptotic expression for $< mm > (k, \omega)$ [30] to the nonasymptotic region [18]. This approach yields $< mm >$ in terms of $w(\ell)$ and $f(\ell)$ [56] where ℓ must now be taken to be k- and ω-dependent. A corresponding analysis was carried further by Hohenberg and Sarkar [43]. A conclusive comparison with experiments above T_λ is not yet possible, both because of the large scatter of the data, and because of

a systematic difference between the data of Vinen and Hurd [11] and those of Tarvin et al. [12] (the latter being somewhat lower than the former).

The situation is more conclusive below T_λ. The striking discrepancy between the asymptotic scaling theory [4, 5, 8] and experiments [10–12] was the weak temperature dependence of the measured linewidth Γ_2 in the hydrodynamic region $k\xi_- < 1$ (Fig. 10). In this region, at sufficiently low frequencies ω (such that first sound is negligible), the dynamic structure factor has the form [57]

$$S(\omega) = \text{const.} \frac{\Gamma_\kappa \omega^2 + \Gamma_\zeta \Omega^2}{(\omega^2 - \Omega^2)^2 + 4\,\Gamma_2^2\, \omega^2} \qquad (25)$$

where $\Omega = c_2 k$ is the second-sound frequency. The linewidth Γ_2 is related to the second-sound damping D_2 according to

$$\Gamma_2 = \frac{1}{2} D_2 k^2 \qquad (26)$$

and consists of two contributions,

$$\Gamma_2 = \Gamma_\kappa + \Gamma_\zeta . \qquad (27)$$

The dependence of $D_2 = 2\,R_2^{eff}\, c_2\, \xi_-$ on $w(\ell)$ and $f(\ell)$ is known [17, 19] from the treatment of second-sound damping (section 6 above). Since the nonuniversal initial conditions $w(\ell_0)$ and $f(\ell_0)$ depend on pressure we have determined these parameters from a (model E) fit to the thermal conductivity data at 22 bar above T_λ [9]. The resulting Γ_2 is compared with the light-scattering data at 23 bar below T_λ in Fig. 10. The excellent agreement resolves the longstanding problem (iii) mentioned in section 1. The essential point is the *decreasing* $f(\ell)$ in the regime $\ell > 10^{-2}$ (corresponding to $T_\lambda - T > 1$ m K) which implies the *increase* of $R_2^{eff} \sim f^{-1/2}$ (Fig. 9). This in turn compensates the predicted [4, 8] temperature dependence $\sim |t|^{-1/3}$ (dashed curve, Fig. 10) resulting only from $c_2 \xi_-$ in D_2, Eq. (21).

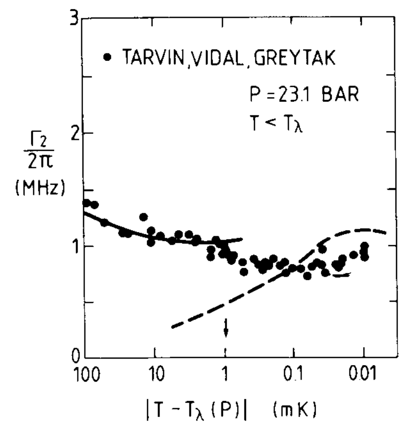

Fig. 10
Comparison between our nonasymptotic theory (full curve) and experimental data [12] for the halfwidth Γ_2 of the dynamic structure factor below T_λ. The dashed curve corresponds to the scaling prediction [4, 5, 8]. The arrow indicates where $k\xi_- = 1$. From Ref. [19].

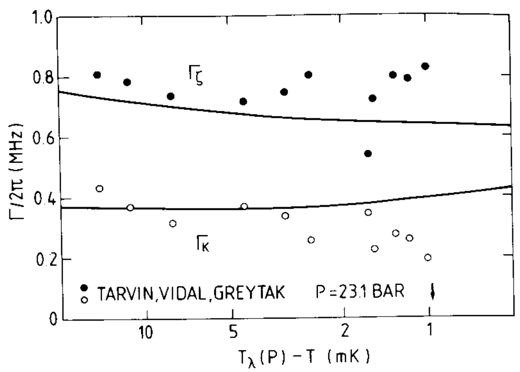

Fig. 11
Comparison between theory (full curves) and experimental data [12] for the transport coefficients Γ_κ and Γ_ζ below T_λ. From Ref. [19].

The separate contributions Γ_κ and Γ_ζ have also been calculated (within model E) as functions of w (ℓ) and f (ℓ) [19]. The comparison with experiment is shown in Fig. 11. This essentially resolves the problem (iv) mentioned in section 1. Nevertheless it should be noted that a more complete model-F study of Γ_κ and Γ_ζ is still lacking.

The more complicated critical region ($k\xi\_ \gtrsim 1$) below T_λ has also been studied [58], and comparison with experiments by means of the nonlinear analysis is in progress.

8 Conclusions and Further Problems

It is now well understood why the observable dynamics of the lambda transition in ^{4}He cannot be described in terms of asymptotic power laws with universal exponents and amplitude ratios. Large crossover effects of genuine dynamic origin govern the critical properties in the experimentally accessible regime. This crossover behavior is well described by the nonlinear flow equations of the field-theoretic renormalization-group approach which permits to treat the entire region between asymptotic criticality and noncritical background behavior.

With a few nonuniversal parameters taken from experiment, the theory explains satisfactorily the long-standing problems mentioned in section 1 and yields excellent agreement with the available data above and below T_λ over nine decades in relative temperature. Additional efforts, both on the theoretical and on the experimental side, are nevertheless desirable in order to provide further confirmation and to make possible a test of the theory in a fully quantitative sense. In particular the problem of the possible breakdown of dynamic scaling at criticality as well as a proper treatment of static quantities in the dynamics below T_λ need further investigation.

The present review was restricted to a discussion of the thermal conductivity, second-sound damping and the low-frequency part of the dynamic structure factor. Equally interesting problems which provide further detailed tests of the renormalization-group theory are the following.

(i) *First sound.* A successful theory of ultrasonic attenuation was recently developed by Ferrell and Bhattacharjee [59]. A brief study in terms of a nonlinear renormalization-group analysis was presented by Miyake [60]. A complete renormalization-group treatment above and below T_λ is still lacking.

(ii) *Viscosity.* Very little is known about a proper theory of the critical viscosity. For a discussion of experiments see [9], for preliminary theoretical remarks see [61].

(iii) *^{3}He- ^{4}He mixtures.* A theory of the thermal conductivity in dilute mixtures was given by Siggia [62] in the spirit of an asymptotic description. A proper nonasymptotic theory which is able to explain the observed behavior of various transport coefficients [63] has not yet been given. Work in this direction is in progress in collaboration with Bhattacharjee [64].

(iv) *Tricritical dynamics.* The asymptotic theory of tricritical dynamics in ^{3}He-^{4}He mixtures by Siggia and Nelson [65] as well as the crossover from λ-line to tricritical behavior needs a reexamination in the spirit of the present review.

(v) *Magnetic systems.* The equivalence of the equations of motions for ^{4}He (model F) with those of planar ($n = 2$) antiferromagnets in a magnet field [6], for example MnF_2 above the spin-flop transition [66], suggests a detailed study of such magnetic systems, including the dynamic crossover behavior near bicritical points [66]. Also a quantitative *nonasymptotic* theory of the dynamics in isotropic ($n = 3$) antiferromagnets, based on the twoloop renormalization-group flow equations [28–30], appears to be an interesting application of a nonlinear renormalization-group analysis.

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Foreword